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# A description of the second mod $p$ cohomology group of a $p$-group

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**Abstract**

We show that any finite mixed Lie algebra in the sense of Lazard, of characteristic $p$ and length 3, is the associated graded of some finite $p$-filtered group. This amounts to calculating the second mod $p$ cohomology group of a $p$-group of $p$-class 2 in terms of the mixed Lie algebra associated to its Frattini filtration. © 1998 Elsevier Science B.V. All rights reserved.

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## 0. Introduction

Let $p$ be a prime number and $G$ be a $p$-group. The *Frattini filtration* (also called lower $p$-series) of $G$ is defined recursively as follows:

$$\Phi_1 G = G; \qquad \Phi_{n+1} G = (\Phi_n G)^p [G, \Phi_n G] \quad \text{for } n \geq 1. \tag{1}$$

Let $\mathrm{gr}_n^\Phi G = \Phi_n G / \Phi_{n+1} G$ be the associated grade of the Frattini filtration: this is a graded $\mathbf{F}_p$-vector space, where $\mathbf{F}_p$ is the field with $p$ elements. But $\mathrm{gr}^\Phi G = \oplus\, \mathrm{gr}_n^\Phi G$ is endowed with two extra structures:

- Bilinear mappings $[\,,\,] : \mathrm{gr}_m^\Phi G \times \mathrm{gr}_n^\Phi G \to \mathrm{gr}_{m+n}^\Phi G$, $i, j \geq 1$, induced by the commutator map in $G$;
- a degree 1 operator $q : \mathrm{gr}_n^\Phi G \to \mathrm{gr}_{n+1}^\Phi G$, induced by "raising to the $p$-th power" in $G$.

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These data are subject to the following relations:
(1) $[\,,]$ is alternating and satisfies the Jacobi identity;
(2) for $x, y \in \mathrm{gr}_n^\Phi G$,

$$q(x+y) = \begin{cases} q(x)+q(y) & \text{if } p>2 \text{ or } n>1, \\ q(x)+q(y)+[x,y] & \text{if } p=2 \text{ and } n=1; \end{cases}$$

(3) for $(x,y) \in \mathrm{gr}_m^\Phi G \times \mathrm{gr}_n^\Phi G$,

$$[q(x),y] = \begin{cases} q([x,y]) & \text{if } p>2 \text{ or } m>1, \\ q([x,y])+[x,[x,y]] & \text{if } p=2 \text{ and } m=1. \end{cases}$$

In other terms, $\mathrm{gr}^\Phi G$ defines a *graded mixed Lie algebra* in the sense of Lazard [13]. If $f: G \to H$ is a group homomorphism, $f$ induces a homomorphism

$$\mathrm{gr}^\Phi f : \mathrm{gr}^\Phi G \to \mathrm{gr}^\Phi H$$

which commutes with $[\,,]$ and $q$.

Those mixed Lie algebras of the form $\mathrm{gr}^\Phi G$ have one further property: they are *generated in degree* 1 (as algebras with operators). Denoting by ($p$-groups) the category of $p$-groups, by $\mathcal{ML}$ the category of finite dimensional mixed Lie algebras and by $\mathcal{GML}$ the full subcategory of mixed Lie algebras generated in degree 1, we therefore have a functor

$$\begin{aligned} \mathrm{gr}^\Phi : (p-\text{groups}) &\to \mathcal{GML} \\ G &\mapsto \mathrm{gr}^\Phi G. \end{aligned}$$

This functor is of great interest in the study of the structure of $p$-groups. It is natural to ask:

**Question.** Is $\mathrm{gr}^\Phi$ essentially surjective?

In other terms, given a (finite dimensional) mixed Lie algebra $A$ generated in degree 1, is there a $p$-group $G$ such that $\mathrm{gr}^\Phi G$ is isomorphic to $A$?

The aim of this paper is to give a partial answer to this question. Let us say that a $p$-group $G$ has *p-class* $\leq n$ if $\Phi_{n+1}G = \{1\}$, and that a mixed Lie algebra $A = (A_n)_{n\geq 1}$ has *length* $\leq n$ if $A_i = 0$ for $i > n$. Let $(p\text{-groups})^{\leq n}$ (resp. $\mathcal{GML}^{\leq n}$) be the full subcategory of ($p$-groups) (resp. $\mathcal{GML}$) consisting of $p$-groups of $p$-class $\leq n$ (resp. algebras of length $\leq n$). Then $\mathrm{gr}^\Phi$ restricts to a functor

$$\mathrm{gr}^\Phi_{\leq n} : (p-\text{groups})^{\leq n} \to \mathcal{GML}^{\leq n}$$

for all $n \geq 1$. Trivially, $\mathrm{gr}^\Phi_{\leq 1}$ is essentially surjective (it is an equivalence of categories!) and it is a (nontrivial) folk result that $\mathrm{gr}^\Phi_{\leq 2}$ is essentially surjective. Our main result is:

**Theorem.** $\mathrm{gr}^{\Phi}_{\leq 3}$ *is essentially surjective.*

Actually we prove a little more than this. As Lazard, we consider the category $(Fil)$ of *p-filtered* groups (Definition 3.1) and prove that the corresponding functor $(Fil)^{\leq 3} \xrightarrow{\mathrm{gr}^{\leq 3}} \mathcal{ML}^{\leq 3}$ is essentially surjective (Theorem 8.5). This says that any mixed Lie algebra of length 3 (generated in degree 1 or not) is the associated graded of some $p$-filtered group.

This paper is organised as follows. Section 1 is devoted to some facts on categories and functors, that we use extensively in this paper. Categories and functors are for us both technically and conceptually convenient. In Section 3, we essentially recall Lazard's elementary theory of $p$-filtered groups and mixed Lie algebras, in the special case of integral filtrations. In Section 4 we introduce our main tool: the functor $s_n$, left adjoint to the functor "truncation at order $n$" on mixed Lie algebras. In Section 6 we relate $s_n$ to group extensions, hence to degree-2 cohomology. In Sections 8 and 9 we state and prove the main theorem. Its original proof by the second author was a difficult cohomological computation (cf. Remark 8.8); here it is replaced by a more elementary group-theoretic proof, whose heart is Lazard's theorem that the Frattini algebra of a free group is free. Finally, in Section 10, we examine how to 'enrich' the mixed algebra structure in order to refine a version of $\mathrm{gr}^{\leq n}$ into an *equivalence of categories*, at least for $n \leq 3$, using structures reminiscent of the secondary cohomology operations of algebraic topology; this section is somewhat inconclusive.

The proof of the key result of this paper Theorem 8.7, as well as certain constructions, are performed here by ad hoc methods, so to say "by hand". It seems likely, however, that all this should be explained by relations between the (co)homology of $p$-groups and that of mixed Lie algebras. We hope to develop the latter theory and investigate these relations in a future paper.

**Notation and terminology.** (1) Throughout the paper, we fix a prime number $p$.

(2) As in [7,8], for a group $G$ and $g,h \in G$, we write $[g,h] = g^{-1}h^{-1}gh$ and $g^h = h^{-1}gh$.

(3) There are two opposite conventions for the use of the term "extension", in group theory and homological algebra. We choose the latter convention. So, in this paper, a group $G$ is an extension of $H$ by $N$ if $N$ is a normal subgroup of $G$ with quotient $H$, or equivalently if there is a short exact sequence $1 \to N \to G \to H \to 1$.

(4) We denote by $H^*(G,A)$ the cohomology of a group $G$ with coefficients in a $G$-module $A$. When $A = \mathbf{Z}/p$ with trivial action, we sometimes abbreviate this notation into $H^*(G)$ or $H^*G$.

The reader should be careful that we use the word "complete" in two different contexts: complete categories (Definition 1.1) and complete $p$-filtered groups (Definition 3.3).

## 1. Representability

In this section, we recall some well known and not-so-well-known facts about categories and functors. The main reference we use is Mac Lane's book [14], except for terminology which is sometimes a little different and more relaxed. Notably we note $\mathrm{Hom}_{\mathscr{C}}(A,B)$ rather than Mac Lane's $\mathscr{C}(A,B)$ for the set of morphisms in a category $\mathscr{C}$. We also prefer the more traditional 'inverse limit' to Mac Lane's "limit".

**Definition 1.1.** A category $\mathscr{C}$ is *complete* if all inverse limits (indexed by a small category) are representable in $\mathscr{C}$.

**Proposition 1.2** (Mac Lane [14, Corollary 2 to Theorem V.2.2]). *A category $\mathscr{C}$ is complete if and only if products and equalisers of two arrows $A \rightrightarrows B$ are representable.*

**Definition 1.3.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $T:\mathscr{C}\to\mathscr{D}$ a functor.

(a) $T$ is *faithful* if, for all $X,Y\in\mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X,Y)\to\mathrm{Hom}_{\mathscr{D}}(T(X),T(Y))$ is injective.

(b) $T$ is *full* if, for all $X,Y\in\mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X,Y)\to\mathrm{Hom}_{\mathscr{D}}(T(X),T(Y))$ is surjective.

(c) $T$ is *essentially surjective* if, for all $Y\in\mathscr{D}$, there exists $X\in\mathscr{C}$ such that $T(X)$ is isomorphic to $Y$.

(d) $T$ is an *equivalence of categories* if there exists $U:\mathscr{D}\to\mathscr{C}$ such that $U\circ T$ and $T\circ U$ are naturally isomorphic respectively to $Id_{\mathscr{C}}$ and $Id_{\mathscr{D}}$.

Traditionally, we say that a functor is *fully faithful* (or *faithfully full*) if it is both full and faithful.

**Proposition 1.4** (Mac Lane [14, Theorem. IV.4.1]). *A functor is an equivalence of categories if and only if it is full, faithful and essentially surjective.*

**Definition 1.5.** Let $\mathscr{C}$ be a category and $\mathscr{X}$ a class of objects of $\mathscr{C}$. The *full subcategory of $\mathscr{C}$ determined by $\mathscr{X}$* is the unique subcategory $\mathscr{D}$ of $\mathscr{C}$ such that:

- $\mathrm{Ob}(\mathscr{D})=\mathscr{X}$;
- for $X,Y\in\mathscr{X}$, $\mathrm{Hom}_{\mathscr{D}}(X,Y)=\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

The inclusion $\mathscr{C}\hookrightarrow\mathscr{D}$ is then fully faithful.

**Definition 1.6.** Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $T:\mathscr{C}\to\mathscr{D}$, $U:\mathscr{D}\to\mathscr{C}$ be two functors.

(a) $T$ is *left adjoint* to $U$ if there exists a natural transformation $\eta : Id_{\mathscr{C}}\to UT$ such that, for all $(X,Y)\in\mathscr{C}\times\mathscr{D}$, the composite map

$$\mathrm{Hom}_{\mathscr{D}}(T(X),Y)\to\mathrm{Hom}_{\mathscr{C}}(UT(X),U(Y))\xrightarrow{\eta^*}\mathrm{Hom}_{\mathscr{C}}(X,U(Y))$$

is bijective. We also say that $U$ is *right adjoint* to $T$. This $\eta$ is called the *unit* of the adjunction.

(b) $T$ is a *section* of $U$ if $UT$ is naturally equivalent to $Id_{\mathscr{C}}$. We also say that $U$ is a *retraction* of $T$.

**Lemma 1.7** (Mac Lane [14, Theorem IV.1.2]). *$T$ is left adjoint to $U$ if and only if there exists a natural transformation $\varepsilon : TU \to Id_{\mathscr{D}}$ such that the composite map*

$$\mathrm{Hom}_{\mathscr{C}}(X, U(Y)) \to \mathrm{Hom}_{\mathscr{D}}(T(X), TU(Y)) \xrightarrow{\varepsilon_*} \mathrm{Hom}_{\mathscr{D}}(T(X), Y)$$

*is bijective. If this is the case, this bijection coincides with that of Definition* 1.6. *The transformation $\varepsilon$ is called the counit of the adjunction.*

**Definition 1.8.** Let $\mathscr{C}$, $\mathscr{D}$, $T$, $U$ be as in Definition 1.6.

(a) $T$ is a *left adjoint-right inverse* of $U$ if it is left adjoint to $U$ and is a section of $U$ through the unit of the adjunction.

(b) $T$ is a *left adjoint-left inverse* of $U$ if it is left adjoint to $U$ and is a retraction of $U$ through the counit of the adjunction.

**Definition 1.9** (Maclane [14, p. 150]). A functor $T : \mathscr{C} \to \mathscr{D}$ *reflects isomorphisms* if, for any morphism $u$ of $\mathscr{C}$, $u$ is an isomorphism $\Leftrightarrow T(u)$ is an isomorphism.

**Lemma 1.10.** *A full functor $T : \mathscr{C} \to \mathscr{D}$ reflects isomorphisms if and only if, for any $X \in \mathscr{C}$, $T^{-1}(Id_{T(X)}) \subseteq Aut_{\mathscr{C}}(X)$.*

**Proof.** Necessity is clear. To see sufficiency, let $X, Y \in \mathscr{C}$ and let $u \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ be such that $T(u)$ is an isomorphism. By the fullness of $T$, there exists $v \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$ such that $uv \in T^{-1}(Id_{T(Y)})$ and $vu \in T^{-1}(Id_{T(X)})$. By the hypothesis, $uv$ and $vu$ are isomorphisms. In particular, $u$ is left and right-invertible, hence is an isomorphism. □

We finally state and prove a proposition which looks classical, but that we have not been able to trace in the literature. As the proof shows, the hypothesis that the base ring is a field is not necessary except for $b$), where we could restrict to projective modules in the general case. Compare [19] and [14, p. 193]. Let (*Sets*) (resp. (*Ab*)) denote the category of sets (resp. of abelian groups).

**Proposition 1.11.** *Let $F$ be a field and $Vect_F$ the category of vector spaces over $F$.*

(a) *Any additive functor $T : Vect_F \to (Ab)$ refines uniquely into an $F$-linear endofunctor of $Vect_F$ followed by the forgetful functor $Vect_F \to (Ab)$.*

(b) *Any $F$-linear endofunctor $T$ of $Vect_F$ is of the form $V \mapsto T_0 \otimes_F V$ for some $T_0 \in Vect_F$.*

(c) *Let $S, T$ be two $F$-linear endofunctors of $Vect_F$, given by $S(V) = S_0 \otimes_F V$ and $T(V) = T_0 \otimes_F V$ after b. Denote by $S_{|(Sets)}, T_{|(Sets)}$ their composition with the forgetful functor $Vect_F \to (Sets)$. Then, in the sequence*

$$\mathrm{Hom}_F(S_0, T_0) \to \mathrm{Hom}_{Vect_F}(S, T) \to \mathrm{Hom}_{(Sets)}(S_{|(Sets)}, T_{|(Sets)}),$$

*both maps are bijective.*

(d) *Let $T$ be an $F$-linear endofunctor of $Vect_F$ and $S$ a subfunctor of $T_{|(Sets)}$ (i.e. $S(V)$ is a subset of $T(V)$ for all $V$), which commutes with finite products. Then $S$ refines into a unique subfunctor of $T$ (i.e. $S(V)$ is a subvector space of $T(V)$ for all $V$).*

**Proof.** (a) Let $V \in Vect_F$. Any $\lambda \in F$ defines an endomorphism of $V$, hence by functoriality an endomorphism $T(\lambda)$ of $T(V)$. For $\lambda, \lambda' \in F$, we clearly have $T(\lambda\lambda') = T(\lambda)T(\lambda')$ and (by additivity) $T(\lambda + \lambda') = T(\lambda) + T(\lambda')$. Hence $T(V)$ inherits a structure of an $F$-vector space, which is obviously natural in $V$.

(b) Set $T_0 = T(F)$; let $V \in Vect_F$. Any $v \in V$ defines an $F$-linear map

$$F \longrightarrow V, \qquad \lambda \mapsto \lambda v$$

hence an $F$-linear map

$$T_0 \to T(V)$$

which is obviously linear in $v$. So we have defined a map

$$V \to \mathrm{Hom}_F(T_0, T(V))$$

which corresponds by adjunction to a map

$$T_0 \otimes_F V \to T(V).$$

For $V = F$, this map is the identity, hence, by additivity of $T$, it is an isomorphism for $V = F^{(I)}$, all $I$. Since every $F$-vector space has a basis, it is an isomorphism for all $V$.

(c) The bijectivity of the first map is clear, as is the injectivity of the second one. To see its surjectivity, let $\eta : S_{|(Sets)} \to T_{|(Sets)}$ be a natural transformation. We want to show that, for all $V$, $\eta_V$ is $F$-linear. It is enough to show that it is additive, as it commutes with the action of $F$ by naturality. This follows from the commutative diagram

$$\begin{tikzcd}
S(V \oplus V) \arrow[r, "\eta_{V\oplus V}"] \arrow[d, "\wr"'] & T(V \oplus V) \arrow[d, "\wr"'] \\
S(V) \oplus S(V) \arrow[r, "\eta_V \times \eta_V"] \arrow[d, "\Sigma_{S(V)}"'] & T(V) \oplus T(V) \arrow[d, "\Sigma_{T(V)}"'] \\
S(V) \arrow[r, "\eta_V"] & T(V)
\end{tikzcd}$$

where, for any $W \in Vect_F$, $\Sigma_W : W \oplus W \to W$ is the addition map, by observing that the big and top square commute and that compositions of the two columns are, respectively, $S(\Sigma_V)$ and $T(\Sigma_V)$.

(d) It is enough to show that, for all $V$, $S(V)$ is closed under addition. This follows from the commutative diagram (analogous to the above):

$$\begin{tikzcd} S(V) \times S(V) \arrow[r, hook] & T(V) \oplus T(V) \\ S(V \oplus V) \arrow[r, hook] \arrow[u, "\wr"] \arrow[d, "S(\Sigma_V)"'] & T(V \oplus V) \arrow[u, "\wr"'] \arrow[d, "T(\Sigma_V)"] \\ S(V) \arrow[r, hook] & T(V). \end{tikzcd}$$

**Remark 1.12.** Part of Proposition 1.11 can be stated as follows: in the sequence

$$Vect_F \xrightarrow{\alpha} \mathrm{Hom}_F(Vect_F, Vect_F) \xrightarrow{\beta} \mathrm{Hom}_{\mathbf{Z}}(Vect_F, (Ab)) \xrightarrow{\gamma} \mathrm{Hom}(Vect_F, (Sets)),$$

where $\mathrm{Hom}_F$ (resp. $\mathrm{Hom}_{\mathbf{Z}}$) denotes $F$-linear (resp. additive) functors, $\alpha(T_0) = T_0 \otimes_F -$ and $\beta, \gamma$ are induced by the forgetful functors, $\alpha$ and $\beta$ are equivalences of categories (a quasi-inverse of $\alpha$ being $T \mapsto T(F)$) and $\gamma$ is fully faithful.

## 2. Group-theoretic relations

In this section, we record four well-known relations that we shall use in the sequel:

$$[a, bc] = [a, c][a, b]^c, \qquad [b, a] = [a, b]^{-1} \tag{2}$$

[7, Hilfsatz II.1.2]

$$\begin{aligned} &(ab)^p \equiv a^p b^p \,(\mathrm{mod}\, \gamma_2(H)^p \gamma_p(H)), \\ &(ab)^2 = a^2 b^2 [a, b^{-1}]^{b^2}, \end{aligned} \tag{3}$$

where $H = \langle a, b \rangle$ and $\gamma_*$ is the descending central series [8, Lemma VIII.1.1]

$$[[a, b^{-1}], c]^b [[b, c^{-1}], a]^c [[c, a^{-1}], b]^a = 1 \tag{4}$$

(Witt's identity) [7, Satz III.1.4]

$$\begin{aligned} &[a, b^p] \equiv [a, b]^p (\mathrm{mod}\, \gamma_2(H)^p \gamma_p(H)), \\ &[a, b^2] = [a, b]^2 [[a, b], b], \end{aligned} \tag{5}$$

where $H$ is as above [8, Lemma VIII.1.1].

## 3. $p$-filtered groups and mixed Lie algebras

**Definition 3.1.** 1. Let $G$ be a group. A *$p$-filtration* on $G$ is a nonincreasing sequence of normal subgroups $(G^{(n)})_{n\geq 1}$ such that

(a) $G^{(1)} = G$;

(b) $[G^{(m)}, G^{(n)}] \subseteq G^{(m+n)}$;

(c) $G^{(n)p} \subseteq G^{(n+1)}$.

A *$p$-filtered group* is a group provided with a $p$-filtration.

2. Let $(G,(G^{(n)}))$ and $(H,(H^{(n)}))$ be two $p$-filtered groups. A *$p$-morphism* between them is a group homomorphism $f: G \to H$ such that $f(G^{(n)}) \subseteq H^{(n)}$ for all $n \geq 1$.

We denote by $(Fil)$ the category whose objects are $p$-filtered groups and morphisms are $p$-morphisms.

**Remark 3.2.** These definitions coincide with Lazard's [13, II.1.1.1 and II.1.2.10], except that we only consider integral $p$-filtrations here.

**Definition 3.3.** A $p$-filtered group $(G,(G^{(n)}))$ is *complete* if the natural homomorphism $G \to \varprojlim G/G^{(n)}$ is an isomorphism. We denote by $(Filc)$ the full subcategory of $(Fil)$ consisting of complete filtered groups.

**Remark 3.4.** The assignment $G \mapsto \hat{G} = \varprojlim G/G^{(n)}$ defines a functor $(Fil) \to (Filc)$ (the completion functor). This functor is evidently a left adjoint-left inverse of the inclusion functor $(Filc) \subset (Fil)$.

**Example 3.5.** For any group $G$, the *Frattini filtration* (1) of $G$ (see introduction) is a $p$-filtration on $G$ [8, Theorem VIII.1.5]. If $G$ is finitely generated, it is complete if and only if it is a pro-$p$-group.

**3.6.** Let $(groups)$ be the category of groups. There is a forgetful functor

$$\begin{aligned} (Fil) &\longrightarrow (groups) \\ (G,(G^{(n)})) &\longmapsto G \end{aligned}$$

**Theorem 3.7.** *The forgetful functor of* 3.6 *has as a left adjoint-right inverse the functor* $\Phi: G \mapsto (G,(\Phi_n G))$, *where* $\Phi_n G$ *is the Frattini filtration of Example* 3.5.

**Proof.** We have to check an equality, for $G \in (groups)$ and $(H,(H^{(n)})) \in (Fil)$:

$$\mathrm{Hom}_{(groups)}(G,H) = \mathrm{Hom}_{(Fil)}((G,(\Phi_n G)),(H,(H^{(n)}))).$$

By forgetting $p$-filtrations, the right set is contained in the left one. Conversely, the axioms (a)–(c) of Definition 3.1 and the definition of the Frattini filtration show that any homomorphism form $G$ to $H$ maps $\Phi_n G$ to $H^{(n)}$ for all $n \geq 1$. Finally, $\Phi$ obviously is also a right inverse of the forgetful functor.

**3.8.** Let $G, H$ be two objects of $(Fil)$ and $r$ an integer $\geq 0$. We define an equivalence relation $\mathscr{R}_r$ on $\mathrm{Hom}_{(Fil)}(G,H)$ as follows:

$$u \equiv v \,(\mathrm{mod}\, \mathscr{R}_r) \Leftrightarrow \text{ for all } i \geq 1 \quad \text{and} \quad g \in G^{(i)}, \qquad v(g)^{-1}u(g) \in H^{(i+r)}.$$

It is readily checked that this indeed defines an equivalence relation, which is compatible with composition of filtered morphisms.

Denote by $(Fil)_r$ the category whose objects are $p$-filtered groups and, for $G, H$ two $p$-filtered groups, $\mathrm{Hom}_{(Fil)_r}(G,H) = \mathrm{Hom}_{(Fil)}(G,H)/\mathscr{R}_r$. We therefore have an inverse system of full "forgetful" functors:

$$\cdots \to (Fil)_r \to \cdots (Fil)_{r-1} \to \cdots \to (Fil)_1 \to (Fil)_0$$

where $(Fil)_0$ is the trivial category where all Hom sets have one element. For æsthetical reasons, one should probably introduce the limit category

$$(\widehat{Fil})$$

with the same objects and, for $G, H$ two $p$-filtered groups,

$$\mathrm{Hom}_{(\widehat{Fil})}(G,H) = \varprojlim \mathrm{Hom}_{(Fil)_r}(G,H).$$

Note that the canonical functors $(Fil) \to (Fil)_r$ induce a "completion" functor $(Fil) \to (\widehat{Fil})$ such that the composite $(Filc) \hookrightarrow (Fil) \to (\widehat{Fil})$ is fully faithful. In practice this refinement will not be necessary here, as we shall almost only consider groups of finite $p$-length.

Proposition 3.9 shows that one does not lose too much information by passing from $(Fil)$ to $(Fil)_1$:

**Proposition 3.9.** *The functor $(Filc) \to (Filc)_1$ (hence also all functors $(Filc) \to (Filc)_r$ and $(Filc)_r \to (Filc)_s$, $r \geq s$) reflects isomorphisms.*

**Proof.** Since the functor is full, it suffices by Lemma 1.10 to show:

**Lemma 3.10.** *Let $G \in (Filc)$ and $u \in End_{(Filc)}(G)$ be such that $u \equiv Id_G \,(\mathrm{mod}\, \mathscr{R}_1)$. Then $u$ is an automorphism of $G$ (as a filtered group).*

It is enough to see that the induced morphism $u_n \in End(G/G^{(n)})$ is bijective for any $n$. We do this by induction on $n$, the case $n = 1$ being trivial. Suppose $u_n$ is surjective. Let $g \in G/G^{(n+1)}$ and $h \in G/G^{(n+1)}$ be such that $u_{n+1}(h) \equiv g \,(\mathrm{mod}\, G^{(n)})$. By assumption on $u$, $u_{n+1}(u_{n+1}(h)^{-1}g) = u_{n+1}(h)^{-1}g$, hence $g \in \mathrm{Im}\, u_{n+1}$ and $u_{n+1}$ is surjective. Let $g \in G$ such that $u(g) \in G^{(n+1)}$. By induction on $n$, $g \in G^{(n)}$. By assumption on $u$, $g^{-1}u(g) \in G^{(n+1)}$ and $g \in G^{(n+1)}$, so $u_{n+1}$ is injective. $\square$

We also have:

**Theorem 3.11.** *The categories* $(Fil)$, $(Filc)$, $(Fil)_r$ *and* $(Filc)_r$ $(r\geq 0)$ *are all complete.*

**Proof.** Clearly, arbitrary products are representable in all four categories; by Proposition 1.2 it suffices to see that equalisers of two morphisms are also representable. Let $\mathscr{C}=(Fil),(Filc),(Fil)_r$ or $(Filc)_r$ $(r\geq 0)$, $G,H\in\mathscr{C}$ and $G \overset{u}{\underset{v}{\rightrightarrows}} H$ be two morphisms in $\mathscr{C}$. For $i\geq 1$, define

$$K^{(i)}=\begin{cases}\{g\in G^{(i)}\mid v(g)^{-1}u(g)\in H^{(i+r)}\} & \text{if } \mathscr{C}=(Fil)_r \text{ or } (Filc)_r,\\ \{g\in G^{(i)}\mid v(g)=u(g)\} & \text{if } \mathscr{C}=(Fil) \text{ or } (Filc),\end{cases}$$

and $K=K^{(1)}$. If $\varphi:M\to G$ is a morphism in $\mathscr{C}$, then $u\circ\varphi=v\circ\varphi$ in $\mathscr{C}$ if and only if $\varphi(M^{(i)})\subseteq K^{(i)}$ for all $i\geq 1$. To show that $(K,(K^{(i)}))$ is an equaliser for $(u,v)$, it is then sufficient to check that $(K^{(i)})$ is a $p$-filtration of $K$. We do it in the case of $(Fil)_r$ and $(Filc)_r$ (the case of $(Fil)$ and $(Filc)$ is similar and easier). To check $[K^{(i)},K^{(j)}]\subseteq K^{(i+j)}$, take $(g,h)\in K^{(i)}\times K^{(j)}$. Setting $\alpha=v(g)^{-1}u(g)$ and $\beta=v(h)^{-1}u(h)$ we have $\alpha\in H^{(i+r)}$, $\beta\in H^{(j+r)}$ and, using (2):

$$\begin{aligned}u([g,h])=[u(g),u(h)]&=[v(g)\alpha,v(h)\beta]\\&=[v(g)\alpha,\beta][v(g),v(h)]^{\beta\alpha}[\alpha,v(h)]^{\beta}\\&=[v(g)\alpha,\beta]v([g,h])^{\beta\alpha}[\alpha,v(h)]^{\beta}\end{aligned}$$

hence

$$v([g,h])^{-1}u([g,h])=[v(g)\alpha,\beta]^{v([g,h])}[v([g,h]),\beta\alpha][\alpha,v(h)]^{\beta}\in H^{(i+j+r)}.$$

To prove that $g^p\in K^{(i+1)}$, we simply use the trivial estimate

$$(xy)^p\equiv x^py^p \pmod{[A,B]}$$

for $(x,y)\in A\times B$ in some ambiant group (with $B$ normal, say). Applying this with $(x,y,A,B)=(v(g),\alpha,H^{(i)},H^{(i+r)})$, we get

$$u(g^p)=u(g)^p=(v(g)\alpha)^p\equiv v(g)^p\alpha^p \pmod{[H^{(i)},H^{(i+r)}]}$$

hence $v(g^p)^{-1}u(g^p)\in H^{(i+r+1)}$, since $[H^{(i)},H^{(i+r)}]\subseteq H^{(2i+r)}$, $\alpha^p\in H^{(i+r+1)}$ and $i+r+1\leq 2i+r$.

To finish the proof in the case $\mathscr{C}=(Filc)$ or $(Filc)_r$, we have to check that $K$ is complete. Since $K^{(i)}\subseteq G^{(i)}$ for all $i$, the filtration $(K^{(i)})$ is separated. Let $(k_i)$ be a sequence of elements of $K$ such that $k_i\equiv k_{i+1} \pmod{K^{(i)}}$ for all $i$. Since $G$ is complete, there exists $g\in G$ such that $g\equiv k_i \pmod{G^{(i)}}$ for all $i$. Taking $i=r+1$, this implies

$$\begin{aligned}v(g)^{-1}u(g)&=v(g_{r+1})^{-1}v(k_{r+1})^{-1}u(g_{r+1})u(k_{r+1})\\&=(v(k_{r+1})^{-1}u(k_{r+1}))^{v(g_{r+1})}v(g_{r+1})^{-1}u(g_{r+1})\end{aligned}$$

where $g_{r+1}=k_{r+1}^{-1}g\in G^{(r+1)}$, hence $g\in K$. $\square$

**Remark 3.12.** In spite of this result, the reader should bear in mind that the natural functors $(Fil) \to (Fil)_r$ and $(Fil)_s \to (Fil)_r$ $(s \geq r)$ are not continuous (do not commute with inverse limits), as the above proof shows. So computing an inverse limit in $(Fil)_r$ does not give in general the same result as in $(Fil)$. For example, there is an exact sequence

$$0 \to \mathbf{Z}_p = \varprojlim_{(Fil)} \mathbf{Z}/p^n \to \varprojlim_{(Fil)_1} \mathbf{Z}/p^n \to \prod_{n \geq 1} \mathbf{Z}/p^n \to 0.$$

**Definition 3.13** (cf. [13, II.1.2]). 1. A *mixed Lie algebra* is a graded $\mathbf{F}_p$-Lie algebra $A = (A_n)_{n \geq 1}$ provided with a degree-1 operator $q : A_n \to A_{n+1}$ satisfying the identities:

(a) for $x, y \in A_n$,

$$q(x+y) = \begin{cases} q(x) + q(y) & \text{if } p > 2 \text{ or } n > 1, \\ q(x) + q(y) + [x,y] & \text{if } p = 2 \text{ and } n = 1; \end{cases}$$

(b) for $(x,y) \in A_m \times A_n$,

$$[q(x), y] = \begin{cases} q([x,y]) & \text{if } p > 2 \text{ or } m > 1, \\ q([x,y]) + [x,[x,y]] & \text{if } p = 2 \text{ and } m = 1. \end{cases}$$

2. Let $(A_n)$ and $(B_n)$ be two mixed Lie algebras. A *morphism* $f$ from $A$ to $B$ is a graded homomorphism $(f_n : A_n \to B_n)_{n \geq 1}$ such that $f_{m+n}([a,b]) = [f_m(a), f_n(b)]$ and $f_{m+1}(q(a)) = q(f_m(a))$ for $(a,b) \in A_m \times A_n$.

We denote by $\mathcal{ML}$ the category whose objects are mixed $p$-Lie algebras and morphisms are morphisms of mixed Lie algebras.

The following lemma follows immediately from the relations:

**Lemma 3.14.** *Let $A$ be a mixed Lie algebra and $a, b$ two homogeneous elements of $A$. If $[a,b] = 0$, then $[a, q(b)] = 0$. In particular, $[a, q(a)] = 0$.*

**Proposition 3.15** (cf. [13, II.1.2.11]). *Let $G$ be a $p$-filtered group, and set $A_n = G^{(n)}/G^{(n+1)}$. Then, the commutator and $p$-power maps in $G$ give $A = (A_n)_{n \geq 1}$ the structure of a mixed Lie algebra* $\operatorname{gr} G$. *If $f : G \to H$ is a morphism of $p$-filtered groups, it induces a morphism $\operatorname{gr} f : \operatorname{gr} G \to \operatorname{gr} H$ of mixed Lie algebras.*

**Proof.** From the definition of a $p$-filtration, the $A_n$ are $p$-elementary abelian, hence $\mathbf{F}_p$-vector spaces. From (2) we deduce that the composite map

$$G^{(m)} \times G^{(n)} \xrightarrow{[,]} G^{(m+n)} \to A_{m+n}$$

induces a bilinear alternating map

$$A_m \times A_n \xrightarrow{[,]} A_{m+n}.$$

Similarly, (3) shows that the composite

$$G^{(n)} \xrightarrow{p} G^{(n+1)} \to A_{n+1}$$

factors as a map

$$A_n \xrightarrow{q} A_{n+1}$$

which satisfies the first identity of Definition 3.13. Finally, the Jacobi identity (resp. the second identity of Definition 3.13) follows from the Witt identity (4) (resp. from (5)).

**Definition 3.16.** (a) Let $A$ be a mixed Lie algebra and $X$ be a subset of $A$. The *subalgebra of $A$ generated by $X$* is the intersection of all mixed Lie subalgebras of $A$ containing $X$.

(b) We say that $A$ is *generated in degree* 1 if the subalgebra of $A$ generated by $A_1$ is $A$.

**Proposition 3.17.** *A mixed Lie algebra $A$ is generated in degree* 1 *if and only if, for all $n > 1$, one has $A_n = [A_1, A_{n-1}] + q(A_{n-1})$.*

**Proof.** The “if” part is obvious, and the “only if” part follows for example from [12, Lemme 2] and the anticommutativity of the product.

**Example 3.18.** If $G$ is provided with its Frattini filtration, we denote by $\mathrm{gr}^{\Phi} G$ the associated mixed Lie algebra and call it the *Frattini algebra* of $G$. We have:

**Corollary 3.19** (to Proposition 3.17). *Let $G$ be a $p$-filtered group. Then* $\mathrm{gr}\, G$ *is generated in degree* 1 *if and only if the filtration of $G$ is its Frattini filtration.*

**3.20.** Proposition 3.15 defines a functor:

$$\mathrm{gr} : (Fil) \to \mathscr{M}\mathscr{L}.$$

We note that this functor factors as a faithful functor

$$\mathrm{gr} : (Fil)_1 \to \mathscr{M}\mathscr{L}$$

where $(Fil)_1$ has been defined in 3.8. This functor will be called the *fundamental functor*. We have the classical

**Proposition 3.21.** *The restriction to $(Filc)_1$ of the fundamental functor reflects isomorphisms.*

## 4. Truncation

**4.1.** Let $n \geq 1$. We define full subcategories $(Fil)^{\leq n}$ and $\mathscr{M}\mathscr{L}^{\leq n}$ of $(Fil)$ and $\mathscr{M}\mathscr{L}$ as follows:

An object $(G, (G^{(m)}))$ of $(Fil)$ is in $(Fil)^{\leq n}$ if $G^{(n+1)} = 1$;

An object $(A_m)$ of $\mathscr{M}\mathscr{L}$ is in $\mathscr{M}\mathscr{L}^{\leq n}$ if $A_m = 0$ for $m > n$.

Note that any group $G \in (Fil)^{\leq n}$ verifies $G^{p^n} = 1$; in particular, it is a $p$-group if finite. Also, any $G \in (Fil)^{\leq n}$ is evidently *complete*.

**4.2.** Clearly, $(Fil)_n^{\leq n} = (Fil)^{\leq n}$. The following concrete description of $(Fil)_{n-1}^{\leq n}$ is especially useful for $n = 2$.

**Lemma 4.3.** *Let* $G, H \in (Fil)^{\leq n}$ *and* $V = \{u \in \mathrm{Hom}_{(Fil)}(G,H) | u \equiv 1 \pmod{\mathscr{R}_{n-1}}\}$. *Then* $V \simeq \mathrm{Hom}(\mathrm{gr}^1 G, \mathrm{gr}^n H)$ *is naturally an* $\mathbf{F}_p$*-vector space, acts freely on* $\mathrm{Hom}_{(Fil)}(G,H)$ *by translation, and* $\mathrm{Hom}_{(Fil)_{n-1}}(G,H) \simeq \mathrm{Hom}_{(Fil)}(G,H)/V$.

**4.4.** The inclusion functor $(Fil)^{\leq n} \to (Fil)$ (*resp.* $\mathscr{ML}^{\leq n} \to \mathscr{ML}$) has a left adjoint-left inverse: the *truncation functor*

$$\begin{aligned} (Fil) &\xrightarrow{T_n} (Fil)^{\leq n} \\ (G, (G^{(m)})) &\mapsto (G/G^{(n+1)}, (G^{(m)}G^{(n+1)}/G^{(n+1)})) \end{aligned}$$

(resp.

$$\mathscr{ML} \xrightarrow{t_n} \mathscr{ML}^{\leq n}$$

$$(A_m) \mapsto (B_m)$$

with

$$B_m = \begin{cases} A_m & \text{if } m \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

the maps $[\,,]$ and $q$ on $B$ being the same as on $A$). It is clear that, for $G \in (Fil)$, the morphism

$$t_n \mathrm{gr}\, G \to \mathrm{gr}(T_n G) \tag{6}$$

induced by $G \to T_n G$ is an isomorphism (although we don't see a category-theoretic reason for this fact!).

**Theorem 4.5** (cf. [13, II.1.2.9]). (a) *Let* $(Vgr)$ *be the category of* $\mathbf{N} - \{0\}$*-graded* $\mathbf{F}_p$*-vector spaces. Then the forgetful functor* $\mathscr{ML} \to (Vgr)$ *has a left adjoint* $F$ *("free algebra on a graded vector space").*

(b) *Let* $X$ *be a set and* $\mathbf{F}_p X$ *the* $\mathbf{F}_p$*-vector space with basis* $X$. *Consider* $\mathbf{F}_p X$ *as a graded* $\mathbf{F}_p$*-vector space concentrated in degree* 1. *Let* $\mathscr{F}X$ *be the free group with basis* $X$. *Then there is a canonical isomorphism*

$$F(\mathbf{F}_p X) \xrightarrow{\sim} \mathrm{gr}^{\Phi} \mathscr{F}X.$$

**Proof.** (a) It is convenient to introduce the category $\mathscr{A}$ of $\mathbf{N} - \{0\}$-graded nonassociative $\mathbf{F}_p$-algebras with one operator. An object of $\mathscr{A}$ consists of a graded vector space $(A_n)_{n \geq 1}$, a bilinear graded product $\cdot : A_i \times A_j \to A_{i+j}$ and an operator of degree 1

$q: A_i \to A_{i+1}$. One does not require the product nor the operator to have any other particular properties. A morphism between two objects of $\mathscr{A}$ is an $\mathbf{F}_p$-linear map of degree 0 which commutes with the two products and $q$s. The forgetful functor of the theorem factors obviously as

$$\mathscr{ML} \xrightarrow{\alpha} \mathscr{A} \xrightarrow{\beta} (Vgr).$$

We shall construct a left adjoint of the two intermediate forgetful functors $\alpha$ and $\beta$; the left adjoint $F$ of the theorem will be their composite.

*Left adjoint to* $\beta$. Given a graded vector space $V = (V_n)_{n \geq 1} \in (Vgr)$, we define an object $A \in \mathscr{A}$ by

$$A_1 = V_1;$$

$$\text{for } n > 1, \quad A_n = V_n \oplus \bigoplus_{i+j=n} A_i \otimes A_j \oplus A_{n-1}.$$

The product $A_i \otimes A_j \to A_{i+j}$ and the operator $q: A_{n-1} \to A_n$ are simply given by the inclusions of the relevant direct summands. There is a degree-0 map of $\mathbf{F}_p$-graded vector spaces $V \to A$ given by the inclusion of the first summand. One checks immediately that this construction is functorial and indeed yields a left adjoint to $\beta$.

*Left adjoint to* $\alpha$. Let $A \in \mathscr{A}$. We associate to it the mixed Lie algebra defined as the quotient of $A$ by the relations $a \cdot a = 0$, $a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b) = 0$ $(a, b, c \in A)$ and the relations of Definition 3.13. Again it is easy to check that this defines a left adjoint to $\alpha$.

(b) The inclusion in $(Vgr)$

$$\mathbf{F}_p X \simeq \mathrm{gr}_1^{\Phi} \mathscr{F} X \hookrightarrow \mathrm{gr}^{\Phi} \mathscr{F} X$$

induces by adjunction a morphism in $\mathscr{ML}$

$$F(\mathbf{F}_p X) \to \mathrm{gr}^{\Phi} \mathscr{F} X.$$

This is the morphism of Theorem 4.5(b). Since it is surjective in degree 1 and $\mathrm{gr}_1^{\Phi} \mathscr{F} X$ is generated in degree 1 (Proposition 3.19), it is surjective. The difficulty is to show it is injective. This is done by Lazard [13, Theorem II.3.2.5] by using the Magnus algebra and the Poincaré–Birkhoff–Witt theorem for mixed Lie algebras (cf. [1, Section 5, Theorem 3] and its proof). □

The following theorem is the central tool of this paper.

**Theorem 4.6.** *The truncation functor* $t_n : \mathscr{ML} \to \mathscr{ML}^{\leq n}$ *has a left adjoint-right inverse* $s_n$.

**Proof.** If $A \in \mathscr{ML}^{\leq n}$ and $F$ is the free functor of Theorem 4.5(a), denote by $\underline{a}$ the image of $a \in A$ in $F(A)$ under the unit (graded vector space-) homomorphism $A \to F(A)$.

Define $s_n(A)$ as the quotient of $F(A)$ by the relations

$$[\underline{a},\underline{b}]=\underline{[a,b]},\qquad q(\underline{a})=\underline{q(a)},$$

for $(a,b)\in A_m\times A_r$ with $m+r\leq n$. We check that $s_n$ is left adjoint to $t_n$. Indeed, let $B\in\mathscr{ML}$ and $\varphi:A\to t_nB$ be a morphism in $\mathscr{ML}^{\leq n}$. Forgetting the algebra structure of $B$ and embedding $t_nB$ into $B$ as a graded subvector space, we extend $\varphi$ into a (graded vector space-) homomorphism

$$\varphi':A\to B$$

which gives by adjunction a mixed Lie algebra homomorphism

$$\varphi'':F(A)\to B$$

where we consider again $B$ as a mixed Lie algebra. For $a,b\in A$, we have

$$\varphi''(\underline{[a,b]})=\varphi([a,b])=[\varphi(a),\varphi(b)]=[\varphi''(\underline{a}),\varphi''(\underline{b})]=\varphi''([\underline{a},\underline{b}])$$

if $\deg(a)+\deg(b)\leq n$ and similarly

$$\varphi''(\underline{q(a)})=\varphi''(q(\underline{a}))$$

if $\deg(a)\leq n-1$. Hence $\varphi''$ factors as a mixed Lie algebra homomorphism $\varphi''':s_nA\to B$. In other words, we have defined a map

$$Hom_{\mathscr{ML}^{\leq n}}(A,t_nB)\to Hom_{\mathscr{ML}}(s_nA,B).$$

On the other hand, the composition

$$A\to F(A)\to s_nA$$

defines a graded vector space homomorphism whose image sits into $t_ns_nA\subseteq s_nA$. By definition of $s_nA$ the resulting linear homomorphism

$$A\to t_ns_nA$$

is a mixed Lie algebra homomorphism (actually an isomorphism). Given now a mixed Lie algebra homomorphism $s_nA\to B$, composition with the former gives a mixed Lie algebra homomorphism $A\to t_nB$, hence a map in the other direction

$$Hom_{\mathscr{ML}}(s_nA,B)\to Hom_{\mathscr{ML}^{\leq n}}(A,t_nB).$$

It is readily checked that composing these maps both ways gives the identity, which completes the proof.

Note that $s_1$ is a special case of $F$. □

**Remarks 4.7.** (1) The truncation functor $T_n:(Fil)\to(Fil)^{\leq n}$ does not preserve all inverse limits (hence does not have a left adjoint). This can be seen by applying it to the pair or maps

$$G\underset{1}{\overset{pr}{\rightrightarrows}}G/G^{(n+1)}$$

for any $p$-filtered group $G$ such that $G^{(n+1)} \neq 1$ and $G^{(n+2)} = 1$. The same counter-example applies in the categories $(Fil)_r$.

(2) The right inverse property of $s_n$ relative to $t_n$ follows formally from the left inverse property of $t_n$ with respect to the inclusion functor $\mathcal{ML}^{\leq n} \xrightarrow{i_n} \mathcal{ML}$.

(3) One could give an *a priori* proof of the existence of $s_n$ by applying Freyd's theorem [14, Theorem V.6.2], i.e. noting that $t_n$ commutes with all inverse limits and checking a "solution set condition". The explicit construction of $s_n$ has the advantage of being suited to computation.

(4) In categorical terms, one can partially explain the construction of $s_n$ by a naturally commutative diagram

$$\begin{tikzcd}
\mathcal{ML}^{\leq n} \arrow[r, shift left=3, "i_n"] \arrow[r, shift right=3, "s_n"'] \arrow[d, "U_n"'] & \mathcal{ML} \arrow[l, "t_n"'] \arrow[d, shift left=2, "U"] \\
(Vgr)^{\leq n} \arrow[r, shift left=3, "\iota_n"] \arrow[r, shift right=3, "\tau_n"'] & (Vgr) \arrow[l, "\tau_n"'] \arrow[u, shift left=2, "F"]
\end{tikzcd}$$

Here $U$ and $U_n$ denote the forgetful functors and $\iota_n, \tau_n$ respectively the inclusion and truncation functors on graded vector spaces. The two main points seem to be that (i) $t_n$ and $\tau_n$ commute with $U$ and $U_n$ and (ii) $\iota_n$ is both left and right adjoint to $t_n$. However these remarks don't completely enlighten the above construction.

(5) From the construction of $s_n$, it follows that
- If $m \leq n$, $s_n(A)_m = A_m$ (right inverse property of $s_n$).
- (for $n > 1$)

$$s_n(A)_{n+1} = \left( \bigoplus_{i+j=n+1} A_i \otimes A_j \oplus A_n \right) \Big/ R,$$

where $R$ is the sub-vector space generated by the relations

$$\begin{array}{lll}
(a \otimes b, 0) = (-b \otimes a, 0), & (a,b) \in A_i \times A_j, & i+j=n+1 \\
(a \otimes a, 0) = 0, & a \in A_{\frac{n+1}{2}} & (\text{if } n+1 \text{ even}) \\
(a \otimes [b,c] + b \otimes [c,a] + c \otimes [a,b], 0) = 0, & (a,b,c) \in A_i \times A_j \times A_k, & i+j+k=n+1
\end{array}$$

and

$$(a \otimes q(b), 0) = \begin{cases} (0, q([a,b]) & \text{if } p>2 \text{ or } i>1 \\ (b \otimes [a,b], q([a,b])) & \text{if } p=2 \text{ and } i=1, \end{cases}$$

$$(a,b) \in A_i \times A_j, i+j=n.$$

- (for $n=1$)

$$s_1(A)_2=\begin{cases} \Lambda^2 A_1 \oplus A_1 & \text{if } p>2,\\ \Gamma^2(A_1) & \text{if } p=2.\end{cases}$$

On the other hand, it becomes more and more complicated to describe $s_n(A)_{n+i}$ in terms of $A$ as $i$ increases. In practice one tries to use e.g. Proposition 4.14.

**4.8.** Instead of shuttling between $\mathcal{ML}$ and $\mathcal{ML}^{\leq n}$, we can shuttle between $\mathcal{ML}^{\leq n}$ and $\mathcal{ML}^{\leq m}$ for $m \leq n$. It is clear that the left adjoint $t_{m,n}$ of the inclusion $\mathcal{ML}^{\leq m} \hookrightarrow \mathcal{ML}^{\leq n}$ is given by $t_m$ restricted to $\mathcal{ML}^{\leq n}$. In view of the left adjoint property of $t_n$, we have:

**Proposition 4.9.** *Let $m \leq n$. The truncation functor $t_{m,n}: \mathcal{ML}^{\leq n} \to \mathcal{ML}^{\leq m}$ has a left adjoint $s_{m,n}$, given by $t_n \circ s_m$.*

**Corollary 4.10.** *Let $B \in \mathcal{ML}^{\leq m}$ and $A = s_m B$. Then, for any $n \geq m$, the counit morphism*

$$s_n(t_n A) \to A$$

*is an isomorphism.*

**Proof.** From the obvious isomorphism of functors

$$t_m = t_{m,n} \circ t_n$$

we get by adjunction an isomorphism of functors

$$s_m = s_n \circ s_{m,n} = s_n \circ t_n \circ s_m. \qquad \square$$

**4.11.** Let $G \in (Fil)$. The inverse of the isomorphism (6)

$$\mathrm{gr}\, T_n G \xrightarrow{\sim} t_n \mathrm{gr}\, G$$

yields by adjunction a morphism

$$s_n(\mathrm{gr}\, T_n G) \to \mathrm{gr}\, G. \tag{7}$$

We have:

**Proposition 4.12.** *For any group $G$ (provided with its Frattini filtration) and any $n \geq 1$, the homomorphism (7)*

$$s_n(\mathrm{gr}^{\Phi} T_n G) \to \mathrm{gr}^{\Phi} G$$

*is surjective. In particular, for $r > n$, $\dim_{\mathbf{F}_p} \mathrm{gr}_r^{\Phi} G \leq \dim_{\mathbf{F}_p} s_n(\mathrm{gr}^{\Phi} T_n G)_r$.*

This follows from Proposition 3.19 and the surjectivity of the map in degree 1.

(Proposition 4.12 can be interpreted as follows: let $\bar{G}$ be a group of $p$-length $n$. For any $G$ such that $T_nG \simeq \bar{G}$, the dimension of the higher Frattini quotients of $G$ is bounded only in terms of $s_n(\mathrm{gr}^{\Phi}\bar{G})$.)

From Theorem 4.5(b) and Corollary 4.10, we also get:

**Proposition 4.13.** *Let $G$ be a free group. Then, for all $n \geq 1$, the homomorphism* (7)

$$s_n(\mathrm{gr}^{\Phi}T_nG) \to \mathrm{gr}^{\Phi}G$$

*is an isomorphism.*

The following proposition and corollary are important in practice.

**Proposition 4.14.** *Let $A \in \mathcal{ML}^{\leq n}$, generated in degree* 1. *Then $s_n(A)$ is generated in degree* 1.

**Proof.** Let $B$ be the mixed subalgebra of $s_n(A)$ generated by $s_n(A)_1 = A_1$. By Remark 4.7 (5) and the hypothesis, $B_m = s_n(A)_m = A_m$ for $m \leq n$. In particular, $t_n(B) = A$. Hence the counit $s_nt_n \to Id$ yields a homomorphism

$$\varphi \in \mathrm{Hom}_{\mathcal{ML}}(s_n(A), B)$$

which is right inverse to the inclusion $B \hookrightarrow s_n(A)$. Therefore this inclusion is surjective and $B = s_n(A)$.

**Corollary 4.15.** *Let $A \in \mathcal{ML}^{\leq n}$, generated in degree* 1. *Then*

$$s_n(A)_{n+1} \simeq \frac{A_1 \otimes A_n \oplus A_n}{R'}$$

*with $R' = R \cap (A_1 \otimes A_n \oplus A_n)$, where $R$ is as in Remark* 4.7(5). *The same holds for $n = 2$, unconditionally on $A$.*

**Proof.** This follows from Propositions 3.17 and 4.14 (the claim for $n = 2$ follows from the anticommutativity of the product).

## 5. An example

Let $p = 2$ and $M$ be the group of 2-class 2 defined by generators and relations by

$$\begin{aligned} M = \langle n_1, n_2, e_1, e_2, f_0, f_1, f_2 \mid & [n_i, f_j] = [e_i, f_j] = f_i^2 = 1, \\ & [n_1, n_2] = [e_1, n_2] = [e_2, n_1] = 1, \quad [e_1, e_2] = f_0f_1f_2, \\ & [e_1, n_1] = n_1^2 = f_0f_1, \ [e_2, n_2] = n_2^2 = f_0f_2, \ e_1^2 = e_2^2 = f_0 \rangle. \end{aligned}$$

For a $p$-group $G$, let $d(G)$ be the minimum number of generators of $G$. Recall [10, Appendix] that $G$ is called *d-maximal* if $d(H) < d(G)$ for any proper subgroup $H$ of $G$. One can show that, up to isomorphism, $M$ is the only 2-class 2$d$-maximal 2-group with $d(M)=4$ such that $s_2(\mathrm{gr}^\Phi M)\notin \mathcal{ML}^{\leq 2}$ (cf. [11, 15]). We have $\Phi(M)=\langle f_0,f_1,f_2\rangle$. Keeping the same notation for generators, $\mathrm{gr}^\Phi M$ has the following presentation:

$$
\begin{aligned}
&\mathrm{gr}^1_\Phi M=\langle e_1,e_2,n_1,n_2\rangle;\\
&\mathrm{gr}^2_\Phi M=\langle f_1,f_2,f_3\rangle;\\
&q(e_1)=q(e_2)=f_0, \qquad q(n_1)=f_0+f_1, \qquad q(n_2)=f_0+f_2;\\
&[e_1,e_2]=f_0+f_1+f_2, \qquad [e_1,n_1]=f_0+f_1, \qquad [e_2,n_2]=f_0+f_2,\\
&[e_1,n_2]=[e_2,n_1]=[n_1,n_2]=0.
\end{aligned}
$$

We shall prove:

**Proposition 5.1** (Kahn [11], Minh [15]). *One has $s_2(\mathrm{gr}^\Phi M)_3\simeq \mathbf{Z}/2$ and $s_2(\mathrm{gr}^\Phi M)_i=0$ for $i>3$.*

**Proof.** Let $A=s_2(\mathrm{gr}^\Phi M)$, so that, as seen above, $t_2(A)=\mathrm{gr}^\Phi M$. Using if necessary Lemma 3.14, we have the following relations in $A$:

$$
\begin{aligned}
&[e_1,f_0]=[e_1,q(e_1)]=0,\\
&[e_2,f_0]=[e_2,q(e_2)]=0,\\
&[n_1,f_0]=[n_1,q(e_2)]=0,\\
&[n_2,f_0]=[n_2,q(e_1)]=0,\\
&[n_1,f_1]=[n_1,f_0+f_1]=[n_1,q(n_1)]=0,\\
&[n_1,f_2]=[n_1,f_0+f_2]=[n_1,q(n_2)]=0,\\
&[n_2,f_1]=[n_2,f_0+f_1]=[n_2,q(n_1)]=0,\\
&[n_2,f_2]=[n_2,f_0+f_2]=[n_2,q(n_2)]=0,\\
&[e_1,f_2]=[e_1,f_0+f_2]=[e_1,q(n_2)]=0,\\
&[e_2,f_1]=[e_2,f_0+f_1]=[e_2,q(n_1)]=0.
\end{aligned}
$$

Also:

$$
\begin{aligned}
[e_1,f_1]&=[e_1,f_0+f_1]=[e_1,q(n_1)]=q([e_1,n_1])+[n_1,[e_1,n_1]]=q(f_0)+q(f_1)\\
&=[e_1,f_0+f_1+f_2]=[e_1,[e_1,e_2]]=q([e_1,e_2])+[e_2,q(e_1)]\\
&=q([e_1,e_2])+[e_2,q(e_2)]=q(f_0)+q(f_1)+q(f_2)
\end{aligned}
$$

hence

$$q(f_2)=0,$$

similarly

$$q(f_1)=0,$$

and so

$$[e_1,f_1]=[e_2,f_2]=q(f_0).$$

This shows that $A_3$ is generated by $g=q(f_0)$, hence $\dim A_3\leq 1$. Moreover, it is clear by Corollary 4.15 that the relations above generate all relations, so that $A_3\neq 0$.

Finally, to see that $A_i = 0$ for $i > 3$, it suffices by Propositions 3.17 and 4.14 to see that $[A_1, A_3] = q(A_3) = 0$. But, for any $x \in A_1 = \mathrm{gr}_1^{\Phi} G$:

$$[x, g] = [x, q(f_0)] = q([x, f_0]) = 0$$

and

$$q(g) = q([e_1, f_1]) = [e_1, q(f_1)] = 0. \qquad \square$$

## 6. Truncation, extensions and cohomology

**6.1.** Let $G \in (Fil)^{\leq n}$ and

(e) $\quad 1 \to A \to \tilde{G} \xrightarrow{\pi} G \to 1$

be an extension. Endow $\tilde{G}$ with the filtration

$$\tilde{G}^{(m)} = \begin{cases} \pi^{-1}(G^{(m)}) & \text{if} m \leq n, \\ A & \text{if} m = n+1, \\ 1 & \text{if} m > n+1. \end{cases}$$

With this filtration, $\tilde{G} \in (Fil)^{\leq n+1}$ if and only if

$$\begin{cases} A^p = 1, \\ [\tilde{G}^{(i)}, \tilde{G}^{(j)}] = 1 & \text{for } i + j = n + 2. \end{cases}$$

In particular, $A$ must be central of exponent $p$. We call such an extension *admissible.*

**Lemma 6.2.** *Let $A$ be an elementary abelian p-group and $H^2_{\mathrm{adm}}(G, A)$ the subset of $H^2(G, A)$ consisting of the classes of admissible extensions (for the p-filtered structure of $G$). Then $H^2_{\mathrm{adm}}(G, A)$ is a subgroup of $H^2(G, A)$, a contravariant functor in $G \in (Fil)^{\leq n}$ and a covariant functor in $A \in$ elementary abelian p-groups.*

**Proof.** The naturality in $A$ and $G$ is obvious. Fixing $G$, the fact that $H^2_{\mathrm{adm}}(G, A)$ is a subgroup of $H^2(G, A)$ follows from Proposition 1.11(d) provided we know it commutes with products. But given two admissible extensions:

$$1 \to A \to \tilde{G} \to G \to 1, \qquad 1 \to B \to \tilde{G}' \to G \to 1,$$

their fibred product over $G$

$$1 \to A \times B \to \tilde{G}'' \to G \to 1$$

is clearly admissible. $\square$

**Proposition 6.3.** *If $G$ is provided with its Frattini filtration, then every central extension of $G$ by an elementary abelian p-group $A$ is admissible.*

This follows from the Witt identity and the definition of $\Phi_n G$ (in terms of mixed Lie algebras, it follows from the relation $[\mathrm{gr}_1^{\Phi} G, A] = 0$, the Jacobi identity and Proposition 3.17). $\square$

**6.4.** Let $\tilde{G}$ be an admissible extension of $G$ with kernel $A$, and $c$ its class in $H^2_{\mathrm{adm}}(G,A)$. We have $t_n(\mathrm{gr}\,\tilde{G}) = \mathrm{gr}\,G$. The counit of the adjunction $(s_n, t_n)$ yields a morphism

$$\theta_{G,A,e} \in \mathrm{Hom}_{\mathcal{M}\mathcal{L}}(s_n(\mathrm{gr}\,G), \mathrm{gr}\,\tilde{G})$$

and in particular a homomorphism

$$\theta_{G,A,e} : s_n(\mathrm{gr}\,G)_{n+1} \to A.$$

Two isomorphic extensions clearly yield the same $\theta_{G,A,e}$. We have therefore defined a map

$$\begin{array}{rcl} H^2_{\mathrm{adm}}(G,A) & \xrightarrow{\theta_{G,A}} & \mathrm{Hom}(s_n(\mathrm{gr}\,G)_{n+1}, A) \\ e & \longmapsto & \theta_{G,A,e} \end{array}$$

which is covariant in $A$ and contravariant in $G$ (as a $p$-filtered group). From this and Proposition 1.11, we deduce:

**Proposition 6.5.** *For any $A, \theta_{G,A}$ is linear. In particular, for $A = \mathbf{Z}/p$ we get a canonical homomorphism*

$$\theta_G : H^2_{\mathrm{adm}}(G, \mathbf{Z}/p) \to (s_n(\mathrm{gr}\,G)_{n+1})^*.$$

*For any $A$, we have $\theta_{G,A} = \theta_G \otimes Id_A$ under the canonical isomorphisms $H^2_{\mathrm{adm}}(G,A) = H^2_{\mathrm{adm}}(G,\mathbf{Z}/p) \otimes A$ and* $\mathrm{Hom}(s_n(\mathrm{gr}\,G)_{n+1}, A) = (s_n(\mathrm{gr}\,G)_{n+1})^* \otimes A$.

*The homomorphism $\theta_G$ defines a natural transformation of $H^2_{\mathrm{adm}}(-,\mathbf{Z}/p)$ to $(s_n(\mathrm{gr}-)_{n+1})^*$ viewed as functors from $(Fil)^{\leq n}$ to $\mathbf{F}_p$-vector spaces.*

We agree to sometimes abbreviate the notation $H^2_{\mathrm{adm}}(G,\mathbf{Z}/p)$ into $H^2_{\mathrm{adm}}(G)$ or $H^2_{\mathrm{adm}}G$.

**Remark 6.6.** Let $G \in (Fil)$. Any homomorphism $\mathrm{gr}^{n+1} G \to \mathbf{Z}/p$ defines by push-out an admissible central extension of $T_n G$ by $\mathbf{Z}/p$. Hence a canonical map

$$(\mathrm{gr}^{n+1} G)^* \to H^2_{\mathrm{adm}}(T_n G, \mathbf{Z}/p)$$

which is seen to be a homomorphism by the same arguments as above. Composing it with $\theta_{T_n G}$, we get a homomorphism

$$(\mathrm{gr}^{n+1} G)^* \to (s_n(\mathrm{gr}\,T_n G)_{n+1})^*.$$

One checks immediately from the definitions that this is the $(n+1)$-st component of the transpose of (7).

**Proposition 6.7.** *Let $G \in (Fil)^{\leq n}$. Then the sequence*

$$H^2_{\mathrm{adm}}(G/G^{(n)}) \xrightarrow{\mathrm{Inf}} H^2_{\mathrm{adm}}(G) \xrightarrow{\theta_G} (s_n(\mathrm{gr}\, G)_{n+1})^*,$$

*where $G/G^{(n)}$ is considered as an object of $(Fil)^{\leq n}$, is exact.*

**Proof.** Let $e \in \operatorname{Ker} \theta_G$, let $1 \to \mathbf{Z}/p \to \tilde{G} \to G \to 1$ be the corresponding extension and let $\tilde{G}^{(n)}$ be the inverse image of $G^{(n)}$ in $\tilde{G}$. The assumption implies that $\tilde{G}^{(n)}$ is elementary abelian and central in $\tilde{G}$. Let $E$ be a complement of $\mathbf{Z}/p$ in $\tilde{G}^{(n)}$, and $H = \tilde{G}/E$. Then $H$ is a central extension of $G/G^{(n)}$ by $\mathbf{Z}/p$, whose class lifts as $e$. Moreover, this extension is admissible since $e$ was. This shows that $\operatorname{Ker} \theta_G \subseteq \operatorname{Im} \operatorname{Inf}$. The opposite inclusion is clear. □

As an application, we get a slight improvement of an upper bound due to P.M. Neumann [17, Corollary 2.2] for $|H^2|$ of a finite $p$-group (the reader should compare our proof with Neumann's):

**Corollary 6.8.** *Let $G$ be a group of order $p^m$ with $d$ generators. Then*:

$$\dim_{\mathbf{F}_p} H^2 G \leq dm - \tfrac{1}{2}\, d(d-1).$$

**Proof.** By induction on the $p$-length $n$ of $G$. For $n = 1$, $G$ is elementary abelian and the inequality is an equality by the known cohomology of elementary abelian $p$-groups. Assume $n \geq 2$. The central extension

$$1 \to \Phi_n G \to G \to G/\Phi_n G \to 1,$$

Propositions 6.3 and 6.7 yield an exact sequence

$$0 \to H^1(\Phi_n G) \to H^2(G/\Phi_n G) \xrightarrow{\mathrm{Inf}} H^2(G) \xrightarrow{\theta_G} (s_n(\mathrm{gr}^{\Phi} G)_{n+1})^*,$$

where the extra injection comes from the Hochschild-Serre spectral sequence and the fact that $\Phi_n G \subseteq \Phi(G)$. This gives the inequality:

$$\dim H^2 G \leq \dim H^2(G/\Phi_n G) - \dim H^1(\Phi_n G) + \dim s_n(\mathrm{gr}^{\Phi} G)_{n+1}.$$

Using Corollary 4.15, we have $\dim s_n(\mathrm{gr}^{\Phi} G)_{n+1} \leq d_n(d+1)$, where $d_n = \dim \Phi_n G$. Moreover, we obviously have $\dim H^1 \Phi_n G = d_n$. The corollary follows. □

We also have:

**Corollary 6.9** (cf. [11]). *Let $M$ be the 2-group of Section 5. Up to isomorphism, there are at most 128 2-groups $G$ such that $T_3 G \simeq M$.*

**Proof.** Let $G$ be such that $T_3 G := G/\Phi_3 G \simeq M$. By Proposition 4.12, $\mathrm{gr}^{\Phi} G$ is a quotient of $s_3(\mathrm{gr}^{\Phi} M)$. By Proposition 5.1, $G$ is of 2-class $\leq 3$ and $\Phi_3 G$ has order 1 or 2, hence is represented by a class $c \in H^2M$. Arguing as above, we have an exact sequence:

$$0 \to H^1(\Phi(G)) \to H^2(M/\Phi(M)) \xrightarrow{\mathrm{Inf}} H^2(M) \xrightarrow{\theta_M} (s_2(\mathrm{gr}^{\Phi} M)_3)^*.$$

The class $c$, if it is $\neq 0$, maps to the nontrivial element $\varepsilon$ of $(s_2(\mathrm{gr}^{\Phi}M)_3)^*$. Since $\dim(M/\Phi(M)) = 4$ and $\dim \Phi(M) = 3$, we have $\dim H^2(M/\Phi(M)) = 10$ and $\dim H^1(\Phi(G)) = 3$. Hence $\theta_M^{-1}(\varepsilon)$ has cardinality either 0 or $2^7 = 128$ (Theorem 8.7 below will show that this cardinality is indeed 128). □

**Convention 6.10**. From now on, we shall restrict to the full subcategories of $(\mathit{Fil})$ (resp. $\mathcal{ML}$) which consist of groups with gradually finite associated graded (resp. gradually finite algebras). We don't change the notations $(\mathit{Fil})$, $\mathcal{ML}$, etc.

## 7. Example: the case $n = 1$

**Theorem 7.1.** *The fundamental functor*

$$\mathrm{gr} : (\mathit{Fil})_1 \to \mathcal{ML}$$

*of* 3.20 *restricts to an equivalence of categories*

$$\mathrm{gr}^{\leq 2} : (\mathit{Fil})_1^{\leq 2} \to \mathcal{ML}^{\leq 2}.$$

**Proof.** We already know $\mathrm{gr}^{\leq 2}$ is faithful; by Proposition 1.4 we have to prove that it is full and essentially surjective. Essential surjectivity follows from

**Proposition 7.2.** *Let $E$ be an elementary abelian group, considered as an object of $(\mathit{Fil})_1$. Then the homomorphism $\theta_E : H^2E \to (s_1(\mathrm{gr}\, E)_2)^*$ is bijective.*

This follows from Propositions 6.3 and 6.7, the description of $s_1(\mathrm{gr}\, E)_2$ in Remark 4.7(5) and the known cohomology of an elementary abelian $p$-group.

It remains to see that $\mathrm{gr}^{\leq 2}$ is full. Let $G, H \in (\mathit{Fil})^{\leq 2}$ and $f : \mathrm{gr}\, G \to \mathrm{gr}\, H$ a homomorphism. We want to see that $f$ comes from a filtered homomorphism from $G$ to $H$. Let $1 \to A_2 \to G \to A_1 \to 1$ and $1 \to B_2 \to H \to B_1 \to 1$ be the central extensions corresponding to the $p$-filtered structures on $G$ and $H$, $c \in H^2(A_1, A_2)$ and $d \in H^2(B_1, B_2)$ the corresponding cohomology classes and $\theta(c) \in \mathrm{Hom}(s_1(A_1)_2, A_2)$, $\theta(d) \in \mathrm{Hom}(s_1(B_1)_2, B_2)$ their images under $\theta_{A_1,A_2}$ and $\theta_{B_1,B_2}$. To say that $f$ is a morphism in $\mathcal{ML}^{\leq 2}$ means that

$$\theta(d) \circ f_1 = f_2 \circ \theta(c) \in \mathrm{Hom}(s_1(A_1)_2, B_2),$$

where $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are the components of $f$. The injectivity of $\theta_{A_1,B_2}$ then implies that

$$f_1^* d = f_{2*} c \in H^2(A_1, B_2).$$

Let $K$ (resp. $L$) be the push-out of $G$ by $f_2$ (resp. the pull-back of $H$ by $f_1$). The equality $f_1^* d = f_{2*} c$ means that there exists an isomorphism $\varphi$ of $K$ onto $L$ which induces the identity on $A_1$ and $B_2$. We then get a morphism $\tilde{f}: G \to H$ in $(Fil)^{\leq 2}$ inducing $f$ by considering the composite:

$$\begin{tikzcd}
1 \arrow[r] & A_2 \arrow[r] \arrow[d, "f_2"'] & G \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & B_2 \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, "\varphi"'] & A_1 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & B_2 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "f_1"'] & 1 \\
1 \arrow[r] & B_2 \arrow[r] & H \arrow[r] & B_1 \arrow[r] & 1.
\end{tikzcd} \qquad (8)$$

□

**Corollary 7.3.** *For $G, H \in (Fil)^{\leq 2}$, $G \simeq H$ if and only if $\operatorname{gr} G \simeq \operatorname{gr} H$.*

This follows from Propositions 3.9 and 3.21.

## 8. The case $n = 2$

**8.1.** For $n > 1$, the fundamental functor

$$\operatorname{gr}^{\leq n+1} : (Fil)_1^{\leq n+1} \to \mathcal{ML}^{\leq n+1}$$

is not full, even when restricted to abelian groups. The reason is simple: if $G_1, G_2 \in (Fil)^{\leq n+1}$ are abelian and $u \in \operatorname{Hom}(G_1, G_2)$, then $\operatorname{gr} u$ certainly respects $\operatorname{Ker} q$. But it must also respect the kernel of "secondary operations", like

$$q^{(2)} : \operatorname{Ker} q \cap \operatorname{gr}^1 G_i \to \operatorname{gr}^3 G_i / q(\operatorname{gr}^2 G_i)$$

coming from the finer structure of $G_1, G_2$ (see 10.2), which is not detected by the functor gr. This makes it easy to construct a counterexample:

**Counterexample 1.** Let $G_1 = \mathbf{Z}/p \times \mathbf{Z}/p \times \mathbf{Z}/p$, with basis $(e_1, e_2, e_3)$; provide $G_1$ with the $p$-filtration $G_1^{(1)} = G_1$, $G_1^{(2)} = \langle e_2, e_3 \rangle$, $G_1^{(3)} = \langle e_3 \rangle$. Let $G_2 = \mathbf{Z}/p \times \mathbf{Z}/p^2$, with basis

$(e_1, e_2)$, provided with the $p$-filtration $G_2^{(1)} = G_2$, $G_2^{(2)} = \langle e_1, e_2^p \rangle$ and $G_2^{(3)} = \langle e_2^p \rangle$. Then $\operatorname{gr} G_1$ and $\operatorname{gr} G_2$ are isomorphic as mixed Lie algebras. But an isomorphism between them cannot come from a homomorphism from $G_1$ to $G_2$, since such a homomorphism would necessarily be an isomorphism by Propositions 3.9 and 3.21. One could concoct similar nonabelian counterexamples.

**8.2.** Here is another argument showing that $\operatorname{gr}^{\leq 3}$ is not full; this argument will help us construct a counterexample with groups provided with their Frattini filtration. Let $\overline{G}$ be of $p$-class 2, provided with its Frattini filtration. Let $c \in H^2(\overline{G})$, $d \in H^2(\overline{G}/\Phi(\overline{G}))$ and let $G, H$ be the central extensions of $\overline{G}$ by $\mathbf{Z}/p$ respectively defined by $c$ and $c + \operatorname{Inf}(d)$. By Proposition 6.7 we have $\operatorname{gr}^{\Phi} G \simeq \operatorname{gr}^{\Phi} H$, but in general $G$ and $H$ will not be isomorphic. More specifically, taking $\overline{G} = \mathbf{Z}/p \times \mathbf{Z}/p^2$:

**Counterexample 2.** Let $G = \mathbf{Z}/p \times \mathbf{Z}/p^3$ and $H = \langle e, f, c \mid [e,f] = f^{p^2} = c,\ e^p = c^p = [e,c] = [f,c] = 1 \rangle$. Then $\operatorname{gr}^{\Phi} G \simeq \operatorname{gr}^{\Phi} H$, but obviously $G \not\simeq H$.

**Remark 8.3.** If $G$ and $H$ are abelian (of any $p$-length), then $\operatorname{Hom}_{(Fil)_1}(G, H) \to \operatorname{Hom}_{\mathscr{ML}}(\operatorname{gr}^{\Phi} G, \operatorname{gr}^{\Phi} H)$ is bijective. This is easily seen by reducing to the case where $G$ and $H$ are cyclic (then both sets have $p$ elements if $|G| \geq |H|$ and 1 otherwise). This shows that one cannot dispense with the noncommutativity of $H$ in Counterexample 2.

**8.4.** A solution to this defect would be to enrich the category of mixed Lie algebras with the extra structure alluded to above. We will not do this here in general, but will give a taste of it in Section 10. But first we shall prove:

**Theorem 8.5.** *The fundamental functor* $\operatorname{gr}^{\leq 3} : (Fil)_1^{\leq 3} \to \mathscr{ML}^{\leq 3}$ *is essentially surjective.*

**Corollary 8.6.** *Let* $\mathscr{GML}^{\leq 3}$ *be the full subcategory of* $\mathscr{ML}^{\leq 3}$ *consisting of mixed Lie algebras generated in degree* 1, $(p-groups)^{\leq 3}$ *the category of p-groups of p-length* $\leq 3$ *and* $\operatorname{gr}^{\Phi}_{\leq 3}$ *the composite functor*

$$(p-groups)^{\leq 3} \xrightarrow{\Phi} (Fil)^{\leq 3} \xrightarrow{\operatorname{gr}^{\leq 3}} \mathscr{ML}^{\leq 3}$$

*viewed as a functor with values in* $\mathscr{GML}^{\leq 3}$ *(by Corollary* 3.19*). Then* $\operatorname{gr}^{\Phi}_{\leq 3}$ *is essentially surjective.*

This follows formally from Theorem 8.5 and Corollary 3.19; but it will actually be proven directly in Section 9.

Theorem 8.5 follows from the more precise statement:

**Theorem 8.7.** *Let* $G \in (Fil)^{\leq 2}$. *Then the map* $\theta_G : H^2_{\mathrm{adm}}(G) \to (s_2(\operatorname{gr} G)_3)^*$ *is surjective.*

**Remark 8.8.** Theorem 8.7 is originally due to the second author for groups provided with their Frattini filtration. The proof we give here is completely different from the original one, that we now summarise for the convenience of the reader. Given a $p$-group $G$ and an elementary abelian subgroup $\Phi$ of $G$ which is central and contained in its Frattini subgroup $\Phi(G)$, there is an exact sequence [16]

$$H^2(K) \xrightarrow{\mathrm{Inf}} H^2(G) \xrightarrow{\Theta} (H^1(K) \otimes H^1(\Phi)) \oplus \beta H^1(\Phi) \to H^3(K),$$

where $K = G/\Phi$ and $\beta$ is the Bockstein homomorphism, refining earlier ones of Iwahori–Matsumoto [9], Thong [18] and Eckmann–Hilton–Stammbach [2, 3], Ganea [4], Gut [5] (in homology); compare especially [9, Proposition 1.1], [5, Theorem 2.3]. (The exactness at $H^2(G)$ is used in a guise in the next section, in case 2 of the proof of Theorem 8.7.) In the special case where $G$ has $p$-class 2 and $\Phi = \Phi(G)$, the map $\Theta$ factors as $\theta_G \circ \pi^*$, where $\pi^*$ is the transpose of the surjection $K \otimes \Phi \oplus \Phi \xrightarrow{\pi} s_2(\mathrm{gr}^{\Phi} G)_3$ of Corollary 4.15. Considering the dual exact sequence

$$H_3(K) \xrightarrow{\rho} H_1(K) \otimes H_1(\Phi) \oplus (\beta H^1(\Phi))^* \to H_2(G) \to H_2(K)$$

and identifying the second group to $K \otimes \Phi \oplus \Phi$, this means that $\mathrm{Im}\,\rho \supseteq \mathrm{Ker}\,\pi$, and the issue is to prove that $\mathrm{Im}\,\rho \subseteq \mathrm{Ker}\,\pi$. But $K$ is elementary abelian, so its homology is “known”, and the latter is checked by an explicit computation (different for $p > 2$ and $p = 2$) on generators of $H_3(K)$.

## 9. Proof of Theorem 8.7

Let us introduce here the following terminology: if $A$ is a mixed Lie algebra of length $n$ and $V$ is an $\mathbf{F}_p$-vector space, we say that a mixed Lie algebra $B$ is a *central extension of $A$ by $V[n+1]$* if $t_n B = A$ and $B_{n+1} = V$ (this is purely for convenience: it is not our intention to develop here the theory of extensions of mixed Lie algebras at large).

The proof proceeds in three steps:

1. *The free case.* Let $G$ be a free group, provided with its Frattini filtration. By Proposition 4.13 the homomorphism from (7)

$$s_2(\mathrm{gr}^{\Phi} T_2 G)_3 \to \mathrm{gr}_3^{\Phi} G$$

is an isomorphism. Dually, the transposed homomorphism

$$(\mathrm{gr}_3^{\Phi} G)^* \to (s_2(\mathrm{gr}^{\Phi} T_2 G)_3)^*$$

is bijective, in particular surjective. By Remark 6.6, this implies that $\theta_{T_2 G}$ is surjective (in fact it is bijective but we do not need that).

2. *The Frattini case.* Let $G$ be a $p$-class 2 $p$-group, provided with its Frattini filtration. Let $E = G/\Phi(G)$, $X$ a basis of $E$, $\mathscr{F}X$ the free group with basis $X$, $\mathscr{F} = \mathscr{F}X/\Phi_3 \mathscr{F}X$ and $\alpha : \mathscr{F} \to G$ the natural surjection. Let $A = \mathrm{gr}^{\Phi} G$, $\tilde{A}$ a central extension of

$A$ by $\mathbf{Z}/p$ [3] and $B$ the pull-back of $\tilde{A}$ by $\mathrm{gr}^{\Phi}\alpha$ (a central extension of $\mathrm{gr}^{\Phi}\mathscr{F}$). By case 1, there is a central extension $H$ of $\mathscr{F}$ by $\mathbf{Z}/p$ such that $\mathrm{gr}^{\Phi} H \simeq B$. Let $F = \mathrm{Ker}\,\alpha$: this is a subgroup of $\Phi(\mathscr{F})$, which is hence central and elementary abelian. As checked inside $B$, its inverse image $\tilde{F}$ in $H$ is still central and elementary abelian. Let $F'$ be a complement of $\mathbf{Z}/p = \mathrm{Ker}(H \to \mathscr{F})$ in $\tilde{F}$ and $\tilde{G} = H/F'$. Then $\tilde{G}$ is a central extension of $G$ by $\mathbf{Z}/p$ such that $\mathrm{gr}^{\Phi}\tilde{G} \simeq \tilde{A}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}/p[3] & \longrightarrow & B & \longrightarrow & \mathrm{gr}^{\Phi}\mathscr{F} & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & {\scriptstyle \mathrm{gr}^{\Phi}\alpha}\big\downarrow & & \\
0 & \longrightarrow & \mathbf{Z}/p[3] & \longrightarrow & \tilde{A} & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & F' \cap & & & & \\
1 & \longrightarrow & Z/p & \longrightarrow & \tilde{F} & \longrightarrow & F & \longrightarrow & 1 \\
 & & \| & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & Z/p & \longrightarrow & H & \longrightarrow & \mathscr{F} & \longrightarrow & 1 \\
 & & \| & & \big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \\
1 & \longrightarrow & Z/p & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & & & \| & & & & \\
 & & & & H/F' & & & &
\end{array}
$$

3. *The general case.* Let $G \in (Fil)^{\leq 2}$, $E = G/G^{(2)}$ and $F$ be a complement of $\Phi(G)$ in $G^{(2)}$. We can find a subgroup $H$ of $G$ such that $G = H \times F$ (direct product); then $H/\Phi(H) \simeq E$ and $\Phi(H) = \Phi(G)$. Let $A = \mathrm{gr}\, G$ and $B = \mathrm{gr}^{\Phi} H$ (a subalgebra of $A$). We have:

$$A_1 = B_1 = E, \qquad A_2 = \Phi(G) \oplus F, \qquad B_2 = \Phi(G).$$

Let $\tilde{A}$ be an extension of $A$ by $\mathbf{Z}/p$ [3]. By restriction to $B$, this defines an extension $\tilde{B}$ of $B$. By case 2, there is a central extension $\tilde{H}$ of $H$ by $\mathbf{Z}/p$ such that $\mathrm{gr}^{\Phi}\tilde{H} \simeq B$. On the other hand, the mixed algebra structure of $\tilde{A}$ gives

- a pairing $[\,,] : E \times (\Phi(G) \oplus F) \to \mathbf{Z}/p$;
- a linear map $q : \Phi(G) \oplus F \to \mathbf{Z}/p$.

Let $K$ be an (abelian) extension of $F$ by $\mathbf{Z}/p$ given by the restriction of $q$ to $F$. The restriction of $[\,,]$ to $E \times F$ defines an action of $E$, hence of $\tilde{H}$, on $K$. Let $C = \mathbf{Z}/p \times \mathbf{Z}/p$ be the kernel of the reduction map

$$\tilde{H} \ltimes K \to H \times F = G$$

and $\Delta$ its diagonal subgroup. Then the group $\tilde{G} = \tilde{H} \ltimes K/\Delta$ is a central extension of $G$ by $\mathbf{Z}/p$ such that $\mathrm{gr}\tilde{G} \simeq \tilde{A}$. $\square$

**Remark 9.1.** In cohomological terms, the proof of case 3 can be interpreted as follows. By the Künneth formula, $H^2G \simeq H^2H \oplus H^1H \otimes H^1F \oplus H^2F$, and $H^2_{\mathrm{adm}}G$ is the subgroup $H^2H \oplus H^1H \otimes H^1F \oplus \beta H^1F \simeq H^2H \oplus E^* \otimes F^* \oplus F^*$, where $\beta$ is the Bockstein map (because the restriction of an admissible extension to $F$ is abelian). On the other hand,

$$s_2(A)_3 \simeq \frac{E \otimes (\Phi(G) \oplus F) \oplus \Phi(G) \oplus F}{R}$$

for $R$ as in Remark 4.7(5), and this is easily seen to be

$$\frac{E \otimes \Phi(G) \oplus \Phi(G)}{R} \oplus E \otimes F \oplus F \simeq s_2(B)_3 \oplus E \otimes F \oplus F.$$

Details are left to the reader.

## 10. Enriching mixed Lie algebras?

**10.1.** We have seen that the functor $(Fil)_1^{\leq 3} \xrightarrow{\mathrm{gr}^{\leq 3}} \mathcal{ML}^{\leq 3}$ is faithful and essentially surjective. This result may be viewed as interesting but worth an improvement (another example of a faithful, essentially surjective functor is the forgetful functor $(groups) \to (sets)$!). It would be nice to refine $\mathrm{gr}^{\leq 3}$ into an *equivalence* of categories $\widetilde{\mathrm{gr}}^{\leq 3}$

$$\begin{tikzcd}
 & \widetilde{\mathcal{ML}}^{\leq 3} \arrow[d] \\
(Fil)_1^{\leq 3} \arrow[ur, "\widetilde{\mathrm{gr}}^{\leq 3}"] \arrow[r, "\mathrm{gr}^{\leq 3}"] & \mathcal{ML}^{\leq 3}
\end{tikzcd}$$

where the category $\widetilde{\mathcal{ML}}^{\leq 3}$ would be an enrichment of $\mathcal{ML}^{\leq 3}$, still of a multilinear nature, and the vertical functor a forgetful functor.

At present we are not able to define a category $\widetilde{\mathcal{ML}}^{\leq 3}$ such that the above conditions are satisfied. The aim of this section, however, is to give ideas on the structures which should presumably be involved in the definition of such a category.

**10.2.** Let $G$ be a $p$-filtered group and $A = \mathrm{gr}\, G$. Let $a \in A_1$ be such that $q(a) = 0 \in A_2$. If $\tilde{a}$ is a lift of $a$ in $G$, we have $\tilde{a}^p \in G^{(3)}$. The image of $\tilde{a}^p$ in $A_3$ does not depend only on $a$, but also on the choice of the lifting $\tilde{a}$. However, changing $\tilde{a}$ into $\tilde{a}b$ with $b \in G^{(2)}$ changes $\tilde{a}^p$ into

$$(\tilde{a}b)^p \equiv \tilde{a}^p b^p \pmod{\gamma_2(H)^p \gamma_p(H)},$$

($H = \langle \tilde{a}, b \rangle$, *cf* (3)) at least for $p > 2$, so that (in this case) the image of $\tilde{a}^p$ in $A_3/q(A_2)$ does not depend on the choice of $\tilde{a}$. In other words, we have defined for $p > 2$ a map

$$q^{(2)} : \mathrm{Ker}\, q \cap A_1 \to A_3/q(A_2).$$

This is an example of an "enrichment" on the structure of the mixed Lie algebra $A = \operatorname{gr} G$. Similarly, let $a, b \in A_1$ be such that $[a,b] = 0 \in A_2$. One sees in the same way as above that, if $\tilde{a}, \tilde{b}$ are lifts of $a, b$ in $G$, the image of the commutator $[\tilde{a}, \tilde{b}] \in G^{(3)}$ in $A_3/[A_1, A_2]$ only depends on $a$ and $b$.

We can go further in this direction. Let $g_1, h_1, \dots, g_m, h_m, k_1, \dots, k_r \in G$ be such that

$$\prod_{i=1}^{m} [g_i, h_i] \prod_{j=1}^{r} k_j^p \in G^{(3)}.$$

It is easily checked that the image of this element in $A_3/[A_1, A_2] + q(A_2)$ only depends on

$$\sum_{i=1}^{m} \bar{g}_i \wedge \bar{h}_i + \sum_{j=1}^{r} \bar{k}_j \in \operatorname{Ker}(\Lambda^2 A_1 \oplus A_1 \xrightarrow{([\,,],q)} A_2),$$

and depends linearly on it (if $p>2$: when $p=2$ one should use squaring and $\operatorname{Ker}(\Gamma^2(A_1) \xrightarrow{q} A_2)$). So we get a well-defined "secondary" homomorphism

$$\operatorname{Ker}(s_1(t_1 A)_2 \to A_2) \xrightarrow{q^{(2)}} \operatorname{Coker}(s_2(t_2 A)_3 \to A_3).$$

Although suggestive, this enriched structure is not enough to define a good category $\widetilde{\mathscr{ML}}^{\le 3}$. For example, for a group $G$ of $p$-class 3 provided with its Frattini filtration, the target group of $q^{(2)}$ is 0 by Proposition 3.19, so this extra structure will certainly not separate isomorphism classes! The answer seems to be that, given an element $x \in \operatorname{Ker}(s_1(t_1 A)_2 \to A_2)$, the element $q^{(2)}(x)$ is in general well-defined in a larger quotient of $A_3$, which depends on $x$: if we represent $x$ as above, it is enough to quotient $A_3$ by

$$V = \sum_{i=1}^{m} [A_2, \bar{g}_i] + \sum_{i=1}^{m} [A_2, \bar{h}_i] + q(A_2),$$

at least for $p>2$. This subgroup of $A_3$ depends on the particular way $x$ is represented in $\operatorname{Ker}(s_1(t_1 A)_2 \to A_2)$, so one could replace it by the intersection of all $V$s corresponding to such representations (we do not know of an intrinsic description of this intersection!). One should investigate, however, the additivity properties of this collection of elements, so that it is not clear what the exact definition of $\widetilde{\mathscr{ML}}^{\le 3}$ should be.

We shall see, nevertheless, that this line of investigation goes in the right direction at least if we restrict to *semi-trivial* mixed Lie algebras of length 3. Let

$$\mathscr{ML}_{\mathrm{st}}^{\le 3}$$

be the full subcategory of $\mathscr{ML}^{\le 3}$ consisting of those algebras $A = (A_1, A_2, A_3)$ such that the maps $A_1 \times A_2 \xrightarrow{[\,,]} A_3$ and $A_2 \xrightarrow{q} A_3$ are identically 0, and let $(Fil)_{1,\mathrm{st}}^{\le 3}$ be the full subcategory of $(Fil)_1^{\leq 3}$ consisting of those $G$ such that $\operatorname{gr} G \in \mathcal{ML}_{\mathrm{st}}^{\leq 3}$. Objects of $(Fil)_{1,\mathrm{st}}^{\leq 3}$ and $\mathcal{ML}_{\mathrm{st}}^{\leq 3}$ are called *semi-trivial.*

**Definition 10.3.** (1) We define a category $\widetilde{\mathcal{ML}}_{\mathrm{st}}^{\leq 3}$ (enriched semi-trivial length 3 mixed Lie algebras) as follows:

(a) An object of $\widetilde{\mathcal{ML}}_{\mathrm{st}}^{\leq 3}$ is a semi-trivial mixed Lie algebra $A$ of length 3, provided with a homomorphism

$$\operatorname{Ker}(s_1(t_1A)_2 \to A_2) \xrightarrow{q^{(2)}} A_3.$$

(b) A morphism of $\widetilde{\mathcal{ML}}_{\mathrm{st}}^{\leq 3}$ is a morphism of semi-trivial mixed Lie algebras $A \xrightarrow{f} B$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Ker}(s_1(t_1A)_2 \to A_2) & \xrightarrow{q^{(2)}} & A_3 \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_3}\downarrow \\
\operatorname{Ker}(s_1(t_1B)_2 \to B_2) & \xrightarrow{q^{(2)}} & B_3
\end{array}$$

commutes, where $f_*$ is the homomorphism induced by $f_1$ and $f_2$.

(2) We define a functor

$$(Fil)_{1,\mathrm{st}}^{\leq 3} \xrightarrow{\widetilde{\operatorname{gr}}_{\mathrm{st}}^{\leq 3}} \widetilde{\mathcal{ML}}_{\mathrm{st}}^{\leq 3}$$

by associating to a semi-trivial length 3 filtered group $G$ its mixed Lie algebra $\operatorname{gr} G$ provided with the enriched structure defined by the $p$-th power and commutator maps as in 10.2.

**Theorem 10.4.** *The functor $\widetilde{\operatorname{gr}}_{\mathrm{st}}^{\leq 3}$ is an equivalence of categories.*

**Proof.** The faithfulness of $\widetilde{\operatorname{gr}}_{\mathrm{st}}^{\leq 3}$ follows from that of $\operatorname{gr}^{\leq 3}$; we have to prove fullness and essential surjectivity.

Let $\bar{G}$ be a $p$-filtered group of length 2 and $A$ an elementary abelian $p$-group. The functor $\widetilde{\operatorname{gr}}_{\mathrm{st}}^{\leq 3}$ induces a map

$$\rho_{\bar{G},A} : \operatorname{Ker}(H^2_{\mathrm{adm}}(\bar{G},A) \xrightarrow{\theta_{\bar{G},A}} \operatorname{Hom}(s_2(\operatorname{gr}\bar{G})_3, A)) \to \operatorname{Hom}(\operatorname{Ker}(s_1(t_1A)_2 \to A_2), A) \tag{9}$$

which is natural in $A$, hence additive by Proposition 1.11. Let $A_i = \operatorname{gr}^i \bar{G}\ (i = 1,2)$.

**Lemma 10.5.** (1) *The diagram*

$$\begin{array}{ccc} H^1(A_2,A) & \xrightarrow{\delta} & H^2(A_1,A) \\ \big\downarrow \wr & & \theta_{A_1,A} \big\downarrow \wr \\ \mathrm{Hom}(A_2,A) & \xrightarrow{\varepsilon_{A_1}^*} & \mathrm{Hom}(s_1(A_1)_2,A) \end{array}$$

*where $\delta$ is the transgression map in the Hochschild–Serre spectral sequence for the extension $1 \to A_2 \to \bar{G} \to A_1 \to 1$, $\varepsilon_{A_1}: s_1(A_1)_2 \to A_2$ is the counit of the adjunction $(s_1,t_1)$ and the left vertical map is the obvious map, anticommutes.*

(2) *The map $\rho_{\bar{G},A}$ of (9) is bijective.*

**Proof.** (1) The explicit description of $\delta$ (see [6, pp. 128–129]) shows that it can be computed as follows: let $f \in H^1(A_2,A)$, viewed as a homomorphism from $A_2$ to $A$. Then $\delta(f) = -f_*c$, where $c \in H^2(A_1,A_2)$ is the class of the extension $G$. It is then enough to check 1) over $Id_{A_2} \in H^1(A_2,A_2)$ (in the case $A=A_2$), in which case it follows tautologically from the definition of $\theta_{A_1,A_2}$. This yields a $\pm$-commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^1(A_2,A) & \xrightarrow{\delta} & H^2(A_1,A) & \longrightarrow & \mathrm{Ker}\,\theta_{\bar{G},A} & \longrightarrow & 0 \\ \big\downarrow \wr & & \theta_{A_1,A} \big\downarrow \wr & & \rho_{\bar{G},A} \big\downarrow & & \\ \mathrm{Hom}(A_2,A) & \xrightarrow{\varepsilon_{A_1}^*} & \mathrm{Hom}(s_1(A_1)_2,A) & \longrightarrow & \mathrm{Hom}(\mathrm{Ker}(s_1(A_1)_2 \to A_2),A) & \longrightarrow & 0 \end{array}$$

hence (2) follows from (1).

We now prove fullness and essential surjectivity in Theorem 10.4 just as in Section 7. Essential surjectivity follows from the essential surjectivity of $\mathrm{gr}^{\le 2}$ (Theorem 7.1) and the surjectivity of $\rho_{\bar{G},A}$. Similarly, let $G,H \in (Fil)^{\le 3}_{1,\mathrm{st}}$, $A = \widetilde{\mathrm{gr}}G$, $B = \widetilde{\mathrm{gr}}H$ and $u: A \to B$ a morphism in $\widetilde{\mathcal{ML}}^{\le 3}_{\mathrm{st}}$. Let $\bar{G} = G/A_3$ and $\bar{H} = H/B_3$. By Theorem 7.1, $t_2(u)$ comes from a homomorphism of filtered groups $\bar{u}: \bar{G} \to \bar{H}$. Let $c \in H^2_{\mathrm{adm}}(\bar{G},A_3)$, $c' \in H^2_{\mathrm{adm}}(\bar{H},B_3)$ be the classes of the extensions $G$ and $H$: they actually sit, respectively, in $\mathrm{Ker}\,\theta_{\bar{G},A_3}$ and $\mathrm{Ker}\,\theta_{\bar{H},B_3}$. The hypothesis that $u$ is a morphism in $\widetilde{\mathcal{ML}}^{\le 3}_{\mathrm{st}}$ implies that $\rho_{\bar{G},B_3}(\bar{u}^*c') = \rho_{\bar{G},B_3}(u_{3*}c) \in \mathrm{Hom}(\mathrm{Ker}(s_1(t_1A)_2 \to A_2),B_3)$. The injectivity of $\rho_{\bar{G},B_3}$ now implies that $\bar{u}^*c' = u_{3*}c \in \mathrm{Ker}\,\theta_{\bar{G},B_3}$, hence a commutative diagram similar to (8). □

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